Assume \( f(x) \) is concave, by definition \( f(ax_i + (1-a)x_i) \geq af(x_i) + (1-a)f(x) \),

\[
f(a_1x_1 + a_2x_2 + asx_s) \geq a_1 f(x_1) + a_2 f(x_2) + as f(x_s)
\]

if \( a_1 + a_2 + as = 1 \)
\( 0 \leq a_1, a_2, as \leq 1 \)

Pf.
\[
f\left(\frac{a_1x_1 + a_2x_2}{(1-as)}\right) \geq a_1 f(x_1) + (1-as) f\left(\frac{a_1x_1 + a_2x_2}{(1-as)}\right)
\]

\[
= a_1 f(x_1) + (1-as) \left[ a_1 \frac{f(x_1)}{(1-as)} + a_2 \frac{f(x_2)}{(1-as)} \right]
\]

\[
= a_1 f(x_1) + as f(x_s) + a_2 f(x_2) + as f(x_s)
\]

We can proceed by induction. For the statement to be true, let say \( f(x) \) is a concave function.

\[
f\left(\sum_{i=1}^{N} a_i x_i\right) \geq \sum_{i=1}^{N} a_i f(x_i)
\]

for \( M+1 \),
\[
f\left(\sum_{i=1}^{M+1} a_i x_i\right) \geq (a_{M+1}) f(x_{M+1}) + \sum_{i=1}^{M} a_i f(x_i)
\]

(by assumption)

\[
> \sum_{i=1}^{M+1} a_i f(x_i)
\]

Statement is true for \( M+1 \), by mathematical induction, the statement is true for all \( M \geq 2 \).
Actually, geometrically, the previous statement is very obvious if we consider a plane curve connecting surface points of a convex function to lie below the function itself.

\[ f(\Sigma ai xi) \]

Jensen inequality is nothing but just the disguise of the lemma studied previously.

\[ \mathbb{E}[f(x)] = \sum p(x) f(x) \]

when \( f(x) \) is concave, note that \( 0 \leq p(x) \leq 1 \) and \( \sum p(x) = 1 \)

then we can treat \( \Sigma p(x) x \) as a

\[ f(\sum p(x) x) = f(\mathbb{E}[x]) \]